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AJ Sadler

Mathematics Methods

4

Student Book

Unit 4

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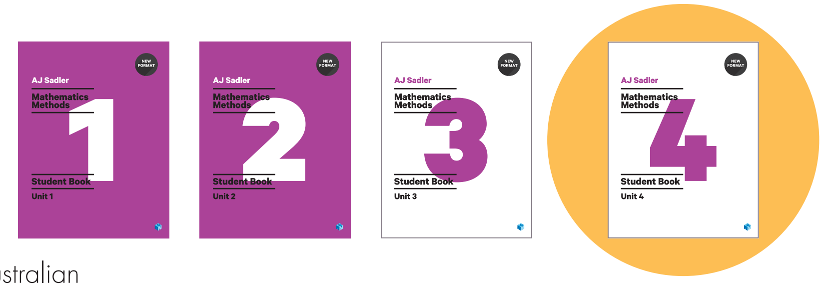
PREFACE



This text targets Unit Four of the West Australian course *Mathematics Methods*, a course that is organised into four units altogether, the first two for year eleven and the last two for year twelve.

This West Australian course, *Mathematics Methods*, is based on the Australian Curriculum Senior Secondary course *Mathematical Methods*. Apart from small changes to wording, the unit fours of these courses are closely aligned. Hence this book would also be suitable for students following unit four of the Australian Curriculum course *Mathematical Methods*.

The book contains text, examples and exercises containing many carefully graded questions. A student who studies the appropriate text and relevant examples should make good progress with the exercise that follows.

The book commences with a section entitled **Preliminary work**. This section briefly outlines work of particular relevance to this unit that students should either already have some familiarity with from the mathematics studied in earlier years, or for which the brief outline included in the section may be sufficient to bring the understanding of the concept up to the necessary level.

As students progress through the book they will encounter questions involving this preliminary work in the **Miscellaneous exercises** that feature at the end of each chapter. These miscellaneous exercises also include questions involving work from preceding chapters to encourage the continual revision needed throughout the unit.

Some chapters commence with a '**Situation**' or two for students to consider, either individually or as a group. In this way students are encouraged to think and discuss a situation, which they are able to tackle using their existing knowledge, but which acts as a forerunner and stimulus for the ideas that follow. Students should be encouraged to discuss their solutions and answers to these situations and perhaps to present their method of solution to others. For this reason answers to these situations are generally not included in the book.

At times in this series of books I have found it appropriate to go a little outside the confines of the syllabus for the unit involved. In this regard readers will find that in this text that when considering sampling I include mention of 'capture – recapture' as an example of sampling, a technique not specifically mentioned in the syllabus, and when considering random sampling I found it appropriate to consider a few simulation activities. When introducing the idea of an interval estimate of a population proportion I also consider the point estimate.

Alan Sadler

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IMPORTANT NOTE

This series of texts has been written based on my interpretation of the appropriate *Mathematics Methods* syllabus documents as they stand at the time of writing. It is likely that as time progresses some points of interpretation will become clarified and perhaps even some changes could be made to the original syllabus. I urge teachers of the *Mathematics Methods* course, and students following the course, to check with the appropriate curriculum authority to make themselves aware of the latest version of the syllabus current at the time they are studying the course.

Acknowledgements

As with all of my previous books I am again indebted to my wife, Rosemary, for her assistance, encouragement and help at every stage.

To my three beautiful daughters, Rosalyn, Jennifer and Donelle, thank you for the continued understanding you show when I am 'still doing sums' and for the love and belief you show.

To the delightfully supportive team at Cengage – I thank you all.

Alan Sadler

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PRELIMINARY WORK

This book assumes that you are already familiar with a number of mathematical ideas from your mathematical studies in earlier years.

This section outlines the ideas which are of particular relevance to Unit Four of the *Mathematics Methods* course and for which some familiarity will be assumed, or for which the brief explanation given here may be sufficient to bring your understanding of the concept up to the necessary level.

Read this 'preliminary work' section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of those concepts up to an appropriate level for this unit. (If you do understand the work but feel somewhat 'rusty' with regards to applying the ideas some of the chapters afford further opportunities for revision as do some of the questions in the miscellaneous exercises at the end of chapters.)

- Chapters in this book will continue some of the topics from this preliminary work by building on the assumed familiarity with the work.
- The miscellaneous exercises that feature at the end of each chapter may include questions requiring an understanding of the topics briefly explained here.

Number

It is assumed that you are familiar with, and competent in the use of, positive and negative numbers, recurring decimals (e.g. 0.666 66…, written $0.\overline{6}$), square roots and cube roots and that you are able to choose levels of accuracy to suit contexts and distinguish between exact values, approximations and estimates.

Numbers expressed with positive, negative and fractional powers should also be familiar to you as should be the following index laws:

$$a^n \times a^m = a^{n+m} \qquad a^n \div a^m = a^{n-m} \qquad a^0 = 1$$

$$a^{-n} = \frac{1}{a^n} \qquad a^{\frac{1}{n}} = \sqrt[n]{a} \qquad (a^n)^m = a^{n \times m}$$

$$(ab)^n = a^n \times b^n \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

Note: The set of numbers that you are currently familiar with is called the set of **real numbers**. We use the symbol $\mathbb{R}$ for this set.

$\mathbb{R}$ contains many subsets of numbers such as the whole numbers, the integers, the prime numbers etc. (If you are also a student of *Mathematics Specialist* you will have encountered numbers beyond this real system. Such considerations are beyond the scope of this unit.)

The absolute value

The absolute value of a number is the distance on the number line that the number is from the origin. The absolute value of x is written $|x|$ and equals x when x is positive, and equals $-x$ when x is negative.
Thus $|3| = 3$, $|-3| = 3$, $|4| = 4$, $|-4| = 4$.

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Algebra

It is assumed that you are already familiar with:

manipulating algebraic expressions, in particular,
- expanding,
- simplifying,
- factorising,

and *solving equations*, in particular, solving:
- linear equations,
- quadratic equations,
- simultaneous equations,
- exponential equations, e.g., $2^x + 3 = 35$,
- trigonometric equations, e.g., $\sin x = 0.5$ for $0 \le x \le 2\pi$.

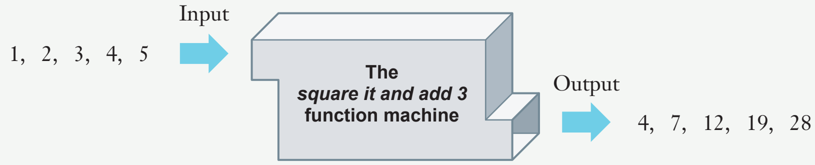
Function

It is assumed that you are familiar with the idea that in mathematics any rule that takes any input value that it can cope with, and assigns to it a particular output value, is called a **function**.

Familiarity with the function notation $f(x)$ is also assumed.

It can be useful at times to consider a function as a machine. A box of numbers (the **domain**) is fed into the machine, a certain rule is applied to each number, and the resulting output forms a new box of numbers, the **range**.

In this way $f(x) = x^2 + 3$, with domain $\{1, 2, 3, 4, 5\}$, could be 'pictured' as follows:



If we are not given a specific domain we assume it to be all the numbers that the function can cope with. This is the function's **natural domain** or **implied domain**.

It is assumed you are particularly familiar with the characteristic equations and graphs of **linear functions**, **quadratic functions** and with the graphs of

$$y = x^3, \qquad y = \sqrt{x} \qquad \text{and} \qquad y = \frac{1}{x},$$

$$y = \sin x, \qquad y = \cos x \qquad \text{and} \qquad y = e^x.$$

It is further assumed that the effect altering the values of a, b, c and d have on the graph of $y = af[b(x - c)] + d$ is something you have previously considered for various functions.

The idea of using the output from one function as the input of a second function should also be familiar to you. In this way we form a **composite function**, also referred to as a **function of a function**.

For example, if $f(x) = x^2$ (the *square it* function)
and $g(x) = x + 3$, (the *add three* function)
then $f(g(x)) = f(x + 3) = (x + 3)^2$ and $g(f(x)) = g(x^2) = x^2 + 3$.

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The exponential function, e^x

In addition to being familiar with the various laws of indices, e.g. $a^n \times a^m = a^{n+m}$, you should also have encountered 'e', which can be defined as $\lim_{n \to \infty}\left[\left(1+\frac{1}{n}\right)^n\right]$, and be familiar with the function $f(x) = e^x$.

- If $y = e^x$ then $\frac{dy}{dx} = e^x$. The exponential function differentiates to itself!
- The constant e ($\approx 2.718\,28$) allows us to describe many situations involving growth (or decay) mathematically.

  Many growth and decay situations involve some variable, say A, growing, or decaying continuously, according to a rule of the form $A = A_0 e^{kt}$
  where A is the amount present at time t,
  A_0 is the initial amount (i.e. the amount present at $t = 0$),
  and k is some constant dependent on the situation.

Summary statistics

Measures of central tendency

The **mean**, the **median** and the **mode** are all measures used to summarise a set of scores. The mean and the median each indicate a 'central score'. The mode is often included in these 'averages' but there is no guarantee that the mode is any 'central' measure.

Measures of spread (or dispersion)

The **range** of a set of scores is the difference between the highest score and the lowest score and gives a simple measure of how widely the scores are spread. Whilst the range is easy to calculate it is determined using just two of the scores and does not take any of the other scores into account. For this reason it is of limited use.

You should be familiar with **variance** and **standard deviation** as more sophisticated measures of dispersion. The variance is found by finding how much each of the scores differs from the mean, squaring these values and finding the average (mean) of the squared values. The standard deviation is the square root of the variance.

Consider the eight scores listed below, for which the mean is 18.

Scores:	12	15	16	16	18	20	22	25
Deviation from mean:	–6	–3	–2	–2	0	+2	+4	+7

Variance of scores $= \dfrac{(-6)^2 + (-3)^2 + (-2)^2 + (-2)^2 + (0)^2 + (2)^2 + (4)^2 + (7)^2}{8}$

$= 15.25$

Standard deviation $= \sqrt{15.25}$ i.e. 3.91 (correct to two decimal places).

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You should be able to determine the mean, median, mode, range, standard deviation and variance of a set of scores when the scores are presented in various forms (e.g. as a list, as a frequency table, as a dot frequency graph etc.), using the ability of your calculator to determine these statistical quantities as appropriate.

$\bar{x}$	$= 18$	⟵ The mean of the scores.
$\sum x$	$= 144$	⟵ The sum of the scores.
$\sum x^2$	$= 2714$	⟵ The sum of the squares of the scores.
$x\sigma_n$	$= 3.90512483$	⟵ The standard deviation of the scores.
$x\sigma_{n-1}$	$= 4.17475405$	⟵ A different standard deviation – see note (2) below.
n	$= 8$	⟵ The number of scores.

(1) The standard deviation is a measure of spread. For most distributions very few, if any, of the scores would be more than three standard deviations from the mean, i.e. the vast majority of the scores (and probably all of them) would lie between $(\bar{x} - 3\sigma)$ and $(\bar{x} + 3\sigma)$.

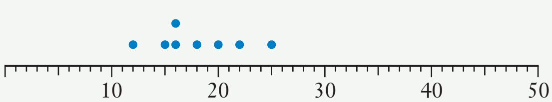
(2) The display shown above has two different standard deviations:
σ_n is the standard deviation of the eight scores.
σ_{n-1} gives an answer a little bigger than σ_n by dividing the sum of the squared deviations by $(n - 1)$ rather than n. This would be used if the eight scores were a sample taken from a larger population and we wanted to use the standard deviation of the sample to estimate the standard deviation of the whole population. Division by $(n - 1)$ rather than n compensates for the fact that there is usually less variation in a small sample than there is in the population itself. If the sample is large, then n will be large and there will be little difference between σ_n and σ_{n-1}.

Change of scale and origin

Consider again the set of eight scores:

12 15 16 16 18 20 22 25

The scores are displayed below left as a dot frequency diagram:



$\bar{x}$	$= 18$
$\sum x$	$= 144$
$\sum x^2$	$= 2714$
$x\sigma_n$	$= 3.90512483$
$x\sigma_{n-1}$	$= 4.17475405$
n	$= 8$

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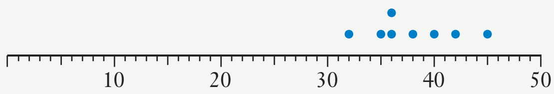
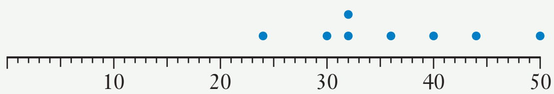
Now suppose we increase all of the scores by 20. This will see them all move 20 places to the right on the dot frequency diagram, i.e. a *change of origin*. With all of the scores increased by 20 we would expect the mean to increase by 20. However, the points are no more, or less, spread out, than they were before. Hence the standard deviation should be unchanged.



$\bar{x}$	= 38
$\sum x$	= 304
$\sum x^2$	= 11674
$x\sigma_n$	= 3.90512483
$x\sigma_{n-1}$	= 4.17475405
n	= 8

Suppose instead we were to multiply all of the original scores by 2, i.e. a *change of scale*. The scores would again all increase in value but would also become more spread out than the original set. We would expect the mean and the standard deviation of this new set of scores to be twice the mean and standard deviation of the original set.



$\bar{x}$	= 36
$\sum x$	= 288
$\sum x^2$	= 10856
$x\sigma_n$	= 7.81024967
$x\sigma_{n-1}$	= 8.34950811
n	= 8

Probability

The probability of something happening is a measure of the likelihood of it happening and is given as a number between zero (no chance of happening) to 1 (certain to happen).

With activities such as rolling a die or flipping a coin, whilst we are unable to consistently predict the outcome of a particular die roll or coin flip, when these activities are repeated a large number of times each has a predictable long-run pattern. For less predictable events the **long-term relative frequency** with which an event occurs is then our best guess at the probability of the event occurring. Probability based on experimental or observed data like this is called **empirical probability**.

You should be familiar with the various 'probability rules' listed on the next page.

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For **complementary** events (A and A′):

$$P(A') = 1 - P(A)$$

For **conditional probability** $(B|A)$:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

For A **and** B $(A \cap B)$:

To determine the probability of A **and** B occurring we multiply the probabilities together, paying due regard to whether the occurrence of one of the events affects the likelihood of the other occurring:

$$P(A \cap B) = P(A) \times P(B|A)$$

If A and B are **independent** events, $P(B|A) = P(B)$ and so

$$P(A \cap B) = P(A) \times P(B)$$

For A **or** B $(A \cup B)$:

To determine the probability of A **or** B occurring we add the probabilities together and then make the necessary subtraction to compensate for the 'double counting of the overlap':

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are **mutually exclusive** events, $P(A \cap B) = 0$ and so

$$P(A \cup B) = P(A) + P(B)$$

Random variables

Suppose a normal fair coin is flipped 3 times. The 8 equally likely outcomes are:

TTT HTT THT TTH THH HTH HHT HHH

If X represents the number of heads obtained, then X can take the values 0, 1, 2, 3.

TTT	HTT THT TTH	THH HTH HHT	HHH
$X = 0$	$X = 1$	$X = 2$	$X = 3$

The value X takes, 0, 1, 2 or 3, depends upon a random selection process.

We call X a **discrete random variable**.

The word **discrete** means 'separate' or 'individually distinct' which is the case here because X can only take the distinct values 0, 1, 2 or 3.

The possible values of a random variable must be numerical.

Discrete random variables commonly occur when we are *counting* events, for example the number of successes in a number of attempts.

In this unit we will extend our understanding of random variables to consider **continuous random variables**. These commonly occur when we are *measuring* something, for example heights, weights, times etc. The variable is not restricted to certain values but can now take any value (usually within certain limits of reasonableness).

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Probability distribution of a discrete random variable

For the random variable X referred to on the previous page, the table below gives the probability associated with each value the variable X can take.

Number of heads (X)	0	1	2	3
Probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

We write $\quad P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8}, \quad P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}.$

This table of probabilities completed for the three flips of a coin situation shows how the total probability of 1 is *distributed* amongst the possible values the variable X can take. The table gives the **probability distribution** for the random variable X.

The possible values the random variable can take must together cover all eventualities without overlap. We say they must be **exhaustive** and **mutually exclusive**.

The sum of the probabilities in a probability distribution must be 1.

From our understanding of probability it also follows that for each value of x,

$$0 \leq P(X = x) \leq 1.$$

For each value the random variable, X, can take, the table assigns the corresponding probability of X taking that value. In mathematics we call a rule or relationship that assigns to each element of one set an element from a second set, a **function**. The pairs of values in the previous table show the **probability function** for the random variable X. We frequently use the notation f(x) to represent a function so we will sometimes use $f(x)$ for $P(X = x)$.

Mean, variance and standard deviation of a discrete random variable

When working with random variables the mean value is sometimes referred to as the **expected value**. For the random variable X, the mean or expected value is sometimes written as E(X). Do not be misled by the use of the word 'expected'. It is not the value we expect to get with one roll of a die, for example, but is instead the number we expect our long-term average to be close to.

For the probability distribution shown at the top of this page, the mean or expected value

$$= 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8}$$
$$= 1.5$$

For the probability distribution shown below:

x	1	2	3	4	5
P($X = x$)	0.1	0.2	0.2	0.4	0.1

the mean or expected value $= 1 \times 0.1 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.4 + 5 \times 0.1$
$= 3.2$

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If the discrete random variable, X, has possible values x_i, with $P(X = x_i) = p_i$ then

$$E(X) = \Sigma(x_i p_i)$$

the summation being carried out over all of the possible values x_i.

If we use the Greek letter, μ, (mu, pronounced myew) to represent $E(X)$, then the **variance**, sometimes written $Var(X)$, is equal to $\Sigma[p_i(x_i - \mu)^2]$.

Prior to the ready availability of calculators, applying this formula could be a tedious process, especially if $E(X)$ was not an integer. In such cases, the alternative formula $Var(X) = E(X^2) - [E(X)]^2$ could be used.

The **standard deviation** is the square root of the variance.

The standard deviation of X is sometimes written $SD(X)$.

The binomial probability distribution

A trial which can be considered to have just two mutually exclusive outcomes, sometimes referred to as *success* (1) and *failure* (0), is called a **Bernoulli trial**. If the probability of success is p then the long term mean = p and variance = $p(1 - p)$.

If a Bernoulli trial is performed repeatedly, with the probability of success in a trial occurring with constant probability, i.e. the trials are **independent**, the distribution that arises by considering the number of successes is called a **binomial distribution**.

If a Bernoulli trial is performed n times, and the probability of success in each trial is p, the probability of exactly x successes in the n trials is

$$^{n}C_x\, p^x(1 - p)^{n-x}$$

The number of trials, n, and the probability of success on each trial, p, are called the **parameters** of the distribution. If we know that a random variable is binomially distributed and the parameters n and p are known, the probability distribution can be completely determined.

If the discrete random variable X is binomially distributed with parameters n and p this is sometimes written as:

$$X \sim b(n, p), \quad X \sim B(n, p), \quad X \sim bin(n, p) \quad \text{or} \quad X \sim Bin(n, p).$$

For example, suppose that each question of a multiple-choice test paper offers five answers, one of which is correct. If a student answers 7 questions by simply guessing which response is correct each time, and if we define the random variable X as how many of these seven questions the student gets correct, we have:

Number of trials = 7,

and P(success, i.e. gets question correct) = 0.2.

Hence $X \sim Bin(7, 0.2)$.

Thus $P(X = 3) = {}^{7}C_3\, 0.2^3\, 0.8^4$

≈ 0.1147

For a binomial distribution involving n trials, with p the probability of success on each trial:

Mean = np

and **Standard deviation** = $\sqrt{np(1-p)}$ or $\sqrt{npq}$ where $q = (1 - p)$, the probability of 'failure' on each trial.

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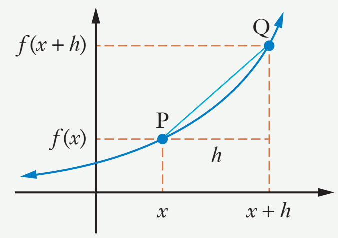
Calculus

Differentiation

It is assumed that you are familiar with the idea of the **gradient**, or *slope*, of a line and in particular that whilst a straight line has the same gradient everywhere, the gradient of a curve varies as we move along the curve.



To find the gradient at a particular point, P, on a curve $y = f(x)$ we choose some other point, Q, on the curve whose x-coordinate is a little more than that of point P.

Suppose P has an x-coordinate of x and Q has an x-coordinate of $(x + h)$.

The corresponding y-coordinates of P and Q will then be $f(x)$ and $f(x + h)$.

Thus the gradient of PQ $= \dfrac{f(x+h) - f(x)}{h}$.

We then bring Q closer and closer to P, i.e. we allow h to tend to zero, and we determine the limiting value of the gradient of PQ.

i.e. Gradient at P $=$ limit of $\dfrac{f(x+h) - f(x)}{h}$ as h tends to zero.

This gives us the **instantaneous rate of change** of the function at P.

The process of determining the **gradient formula** or **gradient function** of a curve is called **differentiation**.

Writing h, the small increase, or increment, in the x coordinate, as δx, and writing $f(x + h) - f(x)$, the small increment in the y coordinate, as δy, we have:

$$\text{Gradient function} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

This **derivative** is written as $\dfrac{dy}{dx}$ and pronounced 'dee y by dee x'.

- If $y = f(x)$ then the derivative of y with respect to x can be written as $\dfrac{dy}{dx}$, $\dfrac{df}{dx}$ or $\dfrac{d}{dx}f(x)$.
- A shorthand notation using a 'dash' may be used for differentiation with respect to x. Thus if $y = f(x)$ we can write $\dfrac{dy}{dx}$ as $f'(x)$ or simply y' or f'.
- Whenever we are faced with the task of finding the gradient formula, gradient function, or derivative of some 'new' function, for which we do not already have a rule, we can simply go back to the basic principle:

$$\text{Gradient at P}(x, f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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Applying this 'limiting chord process' leads to the following results:

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$.

If $y = e^x$ then $\frac{dy}{dx} = e^x$.

If $y = \sin x$ (for x in radians) then $\frac{dy}{dx} = \cos x$.

If $y = \cos x$ (for x in radians) then $\frac{dy}{dx} = -\sin x$.

These facts, together with the rules that follow, allow us to determine the gradient function for many other functions.

With u and v each functions of x, then:

- If $y = u \pm v$, $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$ Sum and difference rules.
- If $y = u \times v$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ The product rule.
- If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ The quotient rule.
- If $y = f(u)$ and $u = g(x)$, $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ The chain rule.
- If $y = [f(x)]^n$, $\frac{dy}{dx} = n[f(x)]^{n-1}f'(x)$ From the chain rule.
- If $y = e^{f(x)}$, $\frac{dy}{dx} = f'(x)e^{f(x)}$ From the chain rule.

Antidifferentiation

You should also be familiar with the idea of **antidifferentiation** which, as its name suggests, is the opposite of differentiation.

E.g. If $\frac{dy}{dx} = ax^n$ then antidifferentiation gives $y = \frac{ax^{n+1}}{n+1} + c$

Remembered as: '*Increase the power by one and divide by the new power.*'

Antidifferentiation is also known as integration, which uses the symbol $\int$.

Hence $$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$$

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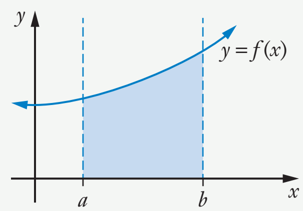
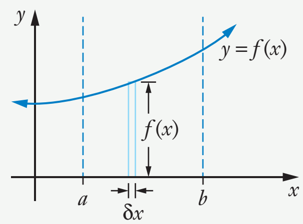
Limit of a sum



The following should remind you of the idea of finding the area under a curve by summing strips of area, and of the fundamental theorem of calculus.

To determine the area between some function $y = f(x)$ and the x axis, from $x = a$ to $x = b$ (see the diagram on the right) we could divide the area into a large number of equal-width strips, each approximately rectangular, and sum the areas of the strips.



One such rectangular strip, of area $y\ \delta x$, is shown in the second diagram on the right.

The more strips, the smaller δx and the greater our accuracy.

If the exact area of the region is A then $$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\ \delta x$$

With a **s**ummation involved we use a 'stretched S' to represent this limit.

We write: $$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\ \delta x = \int_a^b y\ dx$$

A very useful mathematical fact is that this 'limit of a sum' process, called integration, can be determined using antidifferentiation. Indeed this explains why we freely use the same 'stretched S symbol', and the word 'integrate', when we are finding antiderivatives.

Hence to find the area under a curve, the limit of a sum we obtain by considering rectangles can be evaluated using antidifferentiation, a much easier process than summing the areas of many rectangles.

To evaluate $\int_a^b f(x)\, dx$:

(1) Antidifferentiate $f(x)$ with respect to x (and omit the '$+ c$').

(2) Substitute b into your answer from (1).

(3) Substitute a into your answer from (1).

(4) Calculate: (Part (2) answer) – (Part (3) answer).

In this way, evaluating $\int_a^b f(x)\, dx$ gives a specific answer, without any '$+ c$' being involved. Integrals of this form are called **definite integrals**.

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Hence the limit of a sum, which we call a definite integral, can be evaluated using antidifferentiation, because integration and differentiation are opposite processes. This is what the **fundamental theorem of calculus** is all about.

The two boxed results shown below show the opposite nature of this relationship between the definite integral and differentiation. They are the two parts of the **fundamental theorem of calculus**.

$$\int_a^b f'(x)dx = f(b) - f(a) \quad \text{and} \quad \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

From the rule above left we see that integrating the derivative of a function 'gives us the function back'.

and from the rule above right, differentiating the integral of a function 'gives us the function back'.

Applications of calculus

From your previous studies you should be familiar with applying calculus in relation to the following concepts:

Determining the area under a curve and between curves.
Locating turning points and points of inflection.
Optimisation.
Rectilinear motion.
Small changes and marginal rates of change.
Total change from rate of change.

Use of technology

You are encouraged to use your calculator, computer programs and the internet during this unit.

However you should make sure that you can also perform the basic processes such as solving equations, sketching graphs, differentiation, determining definite integrals, without the assistance of such technology when required to do so.

$\frac{d}{dt}(5t^2+6t)$	
	$10\cdot t+6$
$\frac{d}{da}(a^3-3a^2+5)\vert a=3$	
	9

$\int 10xe^{x^2}\,dx$	
	$5\cdot e^{x^2}$
$\int_0^1 8e^{2x}\,dx$	
	$4\cdot(e^2-1)$

Note

The illustrations of calculator displays shown in the book may not exactly match the display from your calculator. The illustrations are not meant to show you exactly what your calculator will necessarily display but are included more to inform you that at that moment the use of a calculator could well be appropriate.

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